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Spacetime properties of ZZ D-branes

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Abstract: We study the tachyon and the RR field sourced by the (m, n) ZZ D-branes in type 0 theories using three methods. We first use the mini-superspace approximation of the closed string wave functions of the tachyon and the RR scalar to probe these fields. These wave functions are then extended beyond the mini-superspace approximation using mild assumptions which are motivated by the properties of the corresponding wave functions in the mini-superspace limit. These are then used to probe the tachyon and the RR field sourced. Finally we study the space time fields sourced by the (m, n) ZZ D-branes using the FZZT brane as a probe. In all the three methods we find that the tension of the (m, n) ZZ brane is mn times the tension of the $(1, 1)$ ZZ brane. The RR charge of these branes is non-zero only for the case of both m and n odd, in which case it is identical to the charge of the $(1, 1)$ brane. As a consistency check we also verify that the space time fields sourced by the branes satisfy the corresponding equations of motion.

Keywords: D-branes, 2D Gravity.

Contents

1. Introduction

The formulation of two dimensional string theories in terms of matrix quantum mechanics has been considered as an example of open/closed string duality. Recent developments in understanding Liouville theory include the discovery of D-brane states in Liouville theory which are localized at the strong coupling region [1], and the subsequent study of the dynamics of these D-branes [2 − 7]. This has given rise to evidence to place this duality along with the list of familiar open/closed string dualities obtained from the AdS/CFT correspondence. Let us consider the bosonic two dimensional string theory. The U(N) matrix quantum mechanics for this theory arises as a world volume theory of the N $(1, 1)$ ZZ branes localized in the strong coupling region. The open string degrees of freedom on the $(1, 1)$ branes are the open string tachyon and a gauge boson. The slope of the inverted harmonic oscillator potential of the matrix quantum is given by the mass of the tachyon, and the presence of the gauge boson restricts the theory to the singlet sector. Liouville theory also contains (m, n) ZZ branes with $m \neq 1, n \neq 1$ [1], the role of these branes in this duality is not clear. These branes contain many more tachyons than the $(1, 1)$ brane and therefore one would expected them to be more unstable. An immediate question one can ask about the properties of these branes is, what is their tension compared to that of the $(1, 1)$ brane. To our knowledge this question has not been addressed in the literature.

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The (m,n) ZZ branes also occur in two dimensional type 0 theories. These theories admit a holographic descriptions in terms of matrix quantum mechanics [8]. To be definite consider type 0B theory, the only two physical fields for generic momenta are the tachyon T in the NS-NS sector and the RR scalar C. The matrix quantum mechanics for this case arises as the world volume theory of unstable $(1,1)$ branes. Consider the D-instanton of this theory, this is a $(1,1)$ brane with Dirichlet boundary conditions in the time direction, it carries a unit RR scalar charge. One can similarly construct the (m,n) D-instanton for $m \neq 1$ and $n \neq 1$. Since the long range fields of type 0B theory is just the tachyon and the RR scalar, this D-instanton must source these target space fields. Again, to elucidate the properties of the (m,n) branes one can ask the question what is the behaviour of the long range closed string fields sourced by these branes and, can one distinguish the index (m,n) by studying these closed string fields. In this paper we address this question by evaluating the space time fields sourced by these branes.

The direct method to find the behaviour of a massless closed string field sourced by a boundary state is to perform the following transform of the one point function of the respective closed string operator on the boundary state.

$$\Psi(z) = \int d^d p K(z,p) \frac{1}{p^2} \langle 0|V_p|B\rangle \,. \tag{1.1}$$

Here $|B\rangle$ refers to the boundary state, and $\langle 0|V_p$ refers to the closed string field. $K(z,p)$ is the wave function of closed string state which are eigen states of momentum p, z stands for the target space coordinate. Evaluating the above transform for the overlap of Dp-brane boundary state with say the graviton of critical string theories gives rise to the standard Coloumb behaviour $1/|z_\perp|^{d_\perp}$, where $z_\perp$ refers to the transverse distance from the Dp-brane and $d_\perp$ is the number of spatial dimensions transverse to the brane. To perform the above transform for the case of the (m,n) D-instanton we face two difficulties: the overlap $\langle 0|V_p|(m,n)\rangle$ has non-trivial momentum dependence, for the case of Dp-brane in critical string theory the momentum dependence is just $\delta(p_\|)$ where $p_\|$ refers to the momentum along the Dp-brane. The second difficulty is that unlike the case of critical string theory where $K(p,z) = e^{ip\cdot z}$ this wave function is at present not known.

In this paper we use three methods to study the behaviour of the closed string field $\Psi(z)$ sourced by the (m,n) D-instanton in type 0B theory. We first use the mini-superspace approximation of the wave-function $K(p,z)$ of the closed string tachyon T and the RR scalar C of type 0B to study the behaviour fields sourced by the (m,n) D-instanton. We find that the tension of an (m,n) brane is proportional to mn and the RR scalar charge of these branes is non zero only for the case of m and n both odd and is equal to that of the $(1,1)$ brane in the mini-superspace approximation. We then postulate the existence of these wave-functions beyond the mini-superspace approximation. We make mild assumptions of the properties of the exact wave functions which are motivated by the properties of the corresponding functions in the mini-superspace limit. In this approximation the wave-functions behave as a superposition of an incoming and an outgoing wave with a relative reflection coefficient far away from the Liouville potential. We assume that this feature still holds beyond the mini-superspace approximation with the reflection coefficient replaced by the exact reflection coefficient. We also make the assumption that the exact wave-functions can be written as a integral transform similar to that of the wave-functions in the mini-superspace limit. With these assumptions we find that behaviour of the tachyon T and the RR scalar C sourced by the (m, n) D-instanton in the region far away from from the Liouville potential is identical to that of the behaviour obtained in the mini-superspace approximation and they satisfy the appropriate equations of motion. Finally we study the behaviour of $\Psi(z)$ using the FZZT brane [9, 10] of Liouville theory as a probe following [11]. The prescription to find the closed string fields $\Psi(z)$ is as follows: the FZZT-ZZ annulus amplitude $Z(m, n|\sigma)$ is written as Laplace transform of the field $\Psi(z)$

$$Z(m, n|\sigma) = \int_0^\infty \frac{dz}{z} e^{-z\cosh \pi\sigma} \Psi(z)\,, \tag{1.2}$$

z labels the target space coordinate. Therefore one can find $\Psi(z)$ by performing the appropriate inverse Laplace transform. Using this approach we find that again the tension of an (m, n) brane is proportional to mn and the RR scalar charge of these branes is non zero only for the case of m and n both odd and is equal to that of the $(1, 1)$ brane. Thus we see that all the three approaches yield the same results. The fact that the RR charge of the (m, n) D-instanton is not proportional to the tension suggests that for $m \neq 1$ and $n \neq 1$, these branes are not stable which can also be seen from the presence of many tachyonic modes in the open string spectrum.

This paper is organized as follows. In section 2 we review some properties of $\mathcal{N} = 1$ Liouville theory, the mini-superspace wave functions, the (m, n) ZZ boundary states and the effective target space action of this theory. In section 3 we find the target space fields in the mini-superspace limit and in section 4 we evaluate them again by postulating the existence of the wave functions $K(p, z)$ beyond the mini-superspace approximation satisfying mild assumptions. Finally in section 5 we use the FZZT probe method study the target space fields. In all these approaches we find that the tension of the (m, n) D-instanton is proportional to mn times the tension of a $(1, 1)$ brane and the RR scalar charge of these branes is non zero only for the case of m and n both odd and is equal to that of the $(1, 1)$ brane.

2. Semi-classical wave-functions and ZZ boundary states

In this section we review some facts concerning the $\mathcal{N} = 1$ Liouville theory which will be used in the computations of the following sections. In section 3 we will perform a semi-classical analysis of the target space field configuration produced by a ZZ brane in type 0B string theory. For that purpose we need to recall the semi-classical wave-functions discussed in [8]. Recall that the Liouville theory is characterized by a background charge $Q = b + 1/b$, in terms of which the central charge is $\hat{c}_L = 1 + 2Q^2$. Together with a free $\hat{c}_T = 1$ system describing the time direction, this gives the correct value $\hat{c} = 10$ if $b = 1$.[1] The semi-classical limit corresponds to $b \to 0$, so we are actually dealing with a strongly coupled world-sheet theory. Nevertheless, it will be instructive to formally consider the

[1] We work with $\alpha' = 1$.

semi-classical limit first. In that limit the $\mathcal{N}=1$ sigma model reduces, in the Ramond sector, to a supersymmetric quantum mechanical system. Due to supersymmetry, the hamiltonian has a doubly degenerate spectrum of wave-functions for non-zero eigenvalue $E=p^2$, $\psi_{p\pm}(z)$ where, if ϕ is the Liouville field and μ the cosmological constant (which we take to be positive from now on), $z=\mu e^{\phi}$ and p is the Liouville momentum. The wave-functions obey:

$$\begin{aligned} \left(z\frac{\partial}{\partial z}+z\right)\Psi_{p+}(z) &= p\Psi_{p-}(z) \\ \left(-z\frac{\partial}{\partial z}+z\right)\Psi_{p-}(z) &= p\Psi_{p+}(z)\,, \end{aligned} \tag{2.1}$$

which are basically supersymmetry transformations, and the eigenvalue equations:

$$\left(-\left(z\frac{\partial}{\partial z}\right)^2 \mp z+z^2-p^2\right)\Psi_{p\pm}(z)=0\,. \tag{2.2}$$

This eigenvalue equation is nothing but that the statement that the full hamiltonian, i.e. the sum of Liouville part and the free, time part, vanishes on physical states.

The term linear in z in (2.2) is due to presence of the fermionic zero-mode bilinear term in the Liouville hamiltonian. Requiring delta-function normalizability, one gets as normalized solutions:

$$\Psi_{p+}(z) = \frac{2}{\Gamma\left(-ip+\frac{1}{2}\right)}\left(\frac{\mu}{2}\right)^{-ip}\sqrt{z}\left(K_{ip-\frac{1}{2}}(z)+K_{ip+\frac{1}{2}}(z)\right) \tag{2.3}$$

$$\Psi_{p-}(z) = \frac{2i}{\Gamma\left(-ip+\frac{1}{2}\right)}\left(\frac{\mu}{2}\right)^{-ip}\sqrt{z}\left(K_{ip-\frac{1}{2}}(z)-K_{ip-\frac{1}{2}}(z)\right). \tag{2.4}$$

For the non-supersymmetric Neveu-Schwarz sector, there no-fermionic zero modes and as a result the linear term $\pm z$ in the hamiltonian is missing in (2.2). The resulting normalized wave function is:

$$\Psi_{p0}(z)=\frac{2}{\Gamma(-ip)}\left(\frac{\mu}{2}\right)^{-ip}K_{ip}(z) \tag{2.5}$$

which satisfies the differential equation:

$$\left(-\left(z\frac{\partial}{\partial z}\right)^2+z^2-p^2\right)\Psi_{p0}(z)=0\,. \tag{2.6}$$

These wave functions are normalized in such a way that for $\phi\to-\infty$ (i.e. $z\to 0$) they are superpositions of an incoming ($e^{ip\phi}$) and outgoing wave ($e^{-ip\phi}$), in the form $\Psi\to e^{ip\phi}+S(p)e^{-ip\phi}$, where the phase S is the reflection coefficient. There are reflection coefficients for R and NS wave-functions $S^R(p)$ and $S^{NS}(p)$, obeying

$$\Psi_{-p\pm}=\pm S^R(p)\Psi_{p\pm}\,,\qquad \Psi_{-p0}=S^{NS}(p)\Psi_{p0}\,. \tag{2.7}$$

Where reflection coefficients are given by;

$$S^R(p)=\left(\frac{\mu}{2}\right)^{2ip}\frac{\Gamma(ip+1/2)}{\Gamma(-ip+1/2)}\,,\qquad S^{NS}(p)=\left(\frac{\mu}{2}\right)^{2ip}\frac{\Gamma(ip)}{\Gamma(-ip)}\,. \tag{2.8}$$

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The boundary states corresponding to (Dirichlet) ZZ branes [12, 13] are labelled by integers (m, n) characterizing the NS(R) degenerate representations of the $\mathcal{N} = 1$ Liouville theory for $m-n = $ even(odd). We will be interested in the related wave-functions, which are the (unnormalized) disc one-point functions of the NSNS and RR ground states respectively. These are nothing but the coefficients relating Ishibashi states, labelled by the continuum momentum p, to Cardy states, labelled by (m, n), and are basically determined by the modular transformation properties of the corresponding characters $\chi_{m.n}$ from the open to the closed channel:

$$\Psi_{NS}(p; m, n) = 2\sinh\left(\pi\frac{mp}{b}\right)\sinh(\pi npb)\left[\frac{\Gamma(1+ipb)\Gamma(1+ip/b)}{(2\pi)^{1/2}\,(-ip)}\left(\frac{\mu}{2}\right)^{-ip/b}\right] \tag{2.9}$$

$$\begin{aligned}\Psi_{RR}(p; m, n) &= -i^{m+n}\,2\sin\left[\pi m\left(\frac{1}{2}+\frac{ip}{b}\right)\right]\sin\left[\pi n\left(\frac{1}{2}+ipb\right)\right]\times\\ &\quad\times\left[\frac{\Gamma(\frac{1}{2}+ipb)\Gamma(\frac{1}{2}+ip/b)}{(2\pi)^{1/2}}\left(\frac{\mu}{2}\right)^{-ip/b}\right].\end{aligned} \tag{2.10}$$

In (2.9) and (2.10) μ actually stands for the renormalized cosmological constant

$$\mu = \mu_0\gamma\left(\frac{1+b^2}{2}\right), \qquad \text{with} \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\,. \tag{2.11}$$

The expressions in (2.9) and (2.10) are exact CFT results, valid for any b. Their mini-superspace limit is obtained taking $b \to 0$ with p/b finite and identifying μ_0 with the semi-classical cosmological constant.

In the case in which b^2 is a rational number, in particular for $b = 1$, that will be the case we will study in this paper, the parametrization of degenerate states with the pair (m, n), with m, n arbitrary positive integers is redundant,[2] since, for example for $b = 1$, all pairs (m, n) with the same value of $m+n$ will correspond to the same degenerate field. Also, the structure of null states is richer compared to the generic irrational case. However, one can show[14], that for $b = 1$, the consequence is simply that the degenerate representations, and therefore the ZZ boundary states, are labelled by $(t, 1)$ with t any positive integer. In any case, we will continue in the following, to adopt the naive (m, n) notation, with the understanding that in the final results of sections 3–5 (m, n) must be specialized to $(t, 1)$.

In the mini-superspace limit, $\Psi_{NS}(p; m, n)$ obey the same reflection relation as Ψ_{p0}, whereas, for $\Psi_{RR}(p; m, n)$, the semi-classical reflection relation is the same as that of Ψ_{p+} times a phase $(-)^{m+n}$.

$$\begin{aligned}\Psi_{NS}(-p; m, n) &= S^{NS}(-p)\Psi_{NS}(p; m, n)\\ \Psi_{RR}(-p; m, n) &= (-)^{m+n}S^{R}(-p)\Psi_{RR}(p; m, n)\,.\end{aligned} \tag{2.12}$$

Notice that these wave-functions do not contain any free parameter corresponding to the position of the D-brane in the ϕ space. In fact, the D-brane is stuck at $\phi \to \infty$, which, due to the linear dependence of the dilaton on ϕ, is the region of strong coupling.

[2] We thank N. Seiberg for pointing this out to us and for bringing to our attention reference [14], where degenerate representations and corresponding boundary states for b^2 rational are discussed.

2.1 Target space fields

The type 0B theory has only two physical fields for generic momenta, the tachyon T in the NSNS sector, and the RR scalar C. The effective action is expected to a classical solution involving a time independent (closed string) tachyon background

$$T(\phi, t) = T(\phi) = \mu e^{\phi}\,. \tag{2.13}$$

together with a linear dilaton and a flat two dimensional metric. The tachyon couples to the RR scalar through the action [8, 15]

$$\frac{1}{8\pi}\int e^{-2T}\, dC \wedge *dC\,. \tag{2.14}$$

The natural field strength associated with C is $F = e^{-T}dC$. The linearized equations of motion and the Bianchi identities for C are given by

$$\begin{aligned} d\left(e^{T}F\right) &= 0 \\ d\left(e^{-T} * F\right) &= 0 \end{aligned} \tag{2.15}$$

that in component reads

$$\begin{aligned} \left(-\frac{\partial}{\partial\phi} + \mu e^{\phi}\right) F_{\phi} &= -\frac{\partial}{\partial t}F_t \\ \left(\frac{\partial}{\partial\phi} + \mu e^{\phi}\right) F_t &= \frac{\partial}{\partial t}F_{\phi}\,. \end{aligned} \tag{2.16}$$

Equations (2.15), for the zero energy, time independent case, become $(\pm z\frac{\partial}{\partial z} + z)F = 0$, whose solutions are $e^{\mp z}$, i.e.:

$$\begin{aligned} F &= e^{-T}dt\,, \\ F &= e^{T}d\phi \end{aligned} \tag{2.17}$$

that are called the electric and magnetic solutions, respectively. Sometimes it can be useful to introduce the field $\chi = e^{-T}C$ that has a canonically normalized kinetic term [8]; in terms of χ the field strength is $F = d\chi + \chi dT$.

A completely equivalent description can be given by introducing the dual scalar $\tilde{C}$ related to C by

$$e^{-T}dC = F = *\tilde{F} = e^{T} * d\tilde{C}\,. \tag{2.18}$$

The RR scalar C couples to D-instantons. D-instantons are D-branes localized both in the (Euclidean) time direction t, and in the Liouville direction ϕ. The corresponding boundary state is the tensor product of the boundary state in the t direction, times the boundary state in the ϕ direction. We have seen that for the latter case there is a supersymmetric generalization of the ZZ boundary state of the bosonic Liouville theory, and the corresponding wave function has been given in (2.9) and (2.10). As remarked in the previous section, while the position of the D-instanton in the time direction is an arbitrary modulus, in the Liouville direction the position is frozen in the strong coupling region $\phi \to \infty$. One of the questions that one can ask is what is the meaning of the (m,n) labels in the ZZ boundary states from the point of view of the target space 0B theory. Various arguments have been given to suggest that only the (1,1) boundary state consistently describes the target space D-instanton, to which the RR scalar C couples minimally. It is still an open problem to give an interpretation to the cases with $(m,n) \neq (1,1)$. This is the question we would like to address in this paper.

3. Target space fields in the minisuperspace limit

In this section we would like to discuss the the following problem: given a ZZ boundary state (times the free, time component part), what is the field configuration, both for RR and NS-NS cases, it produces in (t,ϕ) space-time. In principle, it is clear what we have to do: given the boundary state wave function $\Psi(p;m,n)e^{iEt_0}$ and the wave function corresponding to the closed string state, $\Psi_p(z)e^{iEt}$, we have to evaluate:

$$\int \mathrm{d}p\mathrm{d}E(\Psi_p(z))^* \, e^{-iEt}\frac{1}{p^2+E^2}\left(\Psi(p;m,n)e^{iEt_0}\right). \tag{3.1}$$

Here t_0 is the arbitrary position of the boundary state in the time direction. To simplify the calculation, we can assume a uniform distribution of D-branes in the time direction and integrate over t_0, so that we have:

$$\int \mathrm{d}p\,(\Psi_p(z))^*\frac{1}{p^2}\,(\Psi(p;m,n)). \tag{3.2}$$

In (3.1), the factor $\frac{1}{p^2}$ is the propagator of the canonically normalized field χ. This is the appropriate propagator, since the one-point function is given by a disk computation with one bulk vertex operator insertion. In the vertex operator corresponds to a RR state then it has to be in the $(-3/2,-1/2)$ or in the $(-1/2,-3/2)$ picture. Thus for RR bulk vertex operator in (3.2) evaluates the RR potential. On the other hand if the bulk vertex operator insertion is in the NSNS sector, it has to be in the $(-1,-1)$ picture. Since we are assuming an uniform distribution of D-instantons in the time direction the configuration is T-dual to a (m,n) D0-brane of type 0A theory.

In the familiar case of D-branes in critical string theory, where the wave functions are just plane waves and (p,E) is replaced by the momentum transverse to the brane, the above procedure is well known to reproduce closed string field configurations which solve the linearized equation of motion coming from the target space effective action. In our case the time component of the wave function is still a plane wave, but the Liouville part involves the appropriate wave functions. For the boundary state wave functions, we have the exact expression which are given in (2.9) and (2.10), but for the wave functions of the closed string states we have just the appropriate wave-functions in the mini-superspace limit given in (2.3) and (2.5). A full string theory computation beyond the mini-superspace limit, would need the knowledge of the exact hamiltonian and the corresponding eigen functions, which is missing.

In any case, we will find instructive to analyze the expression (3.2) in the mini-superspace limit $b \to 0$. Therefore, we will take the $b \to 0$ limit in the boundary state wave functions (2.9) and (2.10). Note that, if we had put in (3.1) the full boundary state wave functions, the integral would have been badly divergent in the UV.

Let us begin by evaluating the RR field C produced by a $(1,1)$-brane. The wave functions corresponding to this state are $\Psi_{p\pm}(z)$, but in view of their reflection properties, and of that of Ψ_{RR}, only Ψ_{p+} contributes in the integral of (3.2), which becomes, after taking $b \to 0$ and setting $m = n = 1$ in $\Psi_{RR}(p; m, n)$:

$$\int \mathrm{d}p \sqrt{z} \left(K_{ip-\frac{1}{2}}(z) + K_{ip+\frac{1}{2}}(z) \right) \frac{1}{p^2} \cosh(\pi p) \,. \tag{3.3}$$

Notice that the factor $\Gamma(\frac{1}{2}+ip)$ as well as a p-dependent phase have been canceled between the two wave functions. The resulting integral is an even function of p and is understood to be suitably regulated both in the in the IR and in the UV. We are actually interested in the field strength, which is given by $F = d\chi + \chi dT$. For the background (2.13) and a time independent RR scalar, this is equivalent to the operator $z\frac{\partial}{\partial z} + z$ acting on the integral (3.3). Since the only dependence on z is in the wave function Ψ_{p+}, we can use equations (2.1). The result of this differential operator acting on (3.3) is just to produce one more factor of p in the numerator and to convert $\Psi_{p+}(z)$ into $\Psi_{p-}(z)$. In order to evaluate the resulting p integral it is convenient to use the integral representation for the modified Bessel function [16]:

$$K_\nu(z) = \int_0^\infty \mathrm{d}x\, x^{\nu-1} e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)}. \tag{3.4}$$

From the above integral representation it is clear that $K_\nu(z) = K_{-\nu}(z)$. After applying $z\frac{\partial}{\partial z} + z$ to (3.3), we are therefore led to evaluate the following type of integral:

$$i\sqrt{z} \int_{-\infty}^{\infty} \mathrm{d}p \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}} \int_0^\infty \mathrm{d}\tau\, p \left(\frac{1}{x} - 1\right) e^{ip\ln x} e^{\pi p a} e^{-\tau p^2} e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)} \tag{3.5}$$

where $a = \pm 1$, due to $\cosh \pi p = \frac{e^{\pi p} + e^{-\pi p}}{2}$, and we have used a Schwinger parameterization for $\frac{1}{p^2}$. The integral can be regularized as usual by cutting off the domain of τ-integration. The p-integral in (3.5), being gaussian, can be readily performed. The resulting expression is proportional to:

$$\sqrt{z} \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}} \int_0^\infty \mathrm{d}y A(x) e^{-\frac{A^2(x)}{4} y^2} \left(\frac{1}{x} - 1\right) e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)}, \tag{3.6}$$

where we have changed integration variable from τ to $y = 1/\sqrt{\tau}$ and defined $A(x) = \ln x - i\pi a$. The y-integral is gaussian for those values of x for which $\mathrm{Re}A^2 < 0$. The idea is then to first cut-off the y-integral, deform the x-integration contour in a region where we can remove the cut-off, so that the y-integral becomes gaussian.

Let us describe the resulting x-integration contour: it will include the region on the positive real axis where $\ln x^2$ is sufficiently large, say $x > R$,[3] together with $[0, 1/R]$. In the

[3] As we will see the final result will be independent on this value, thus we do not need to specify it.

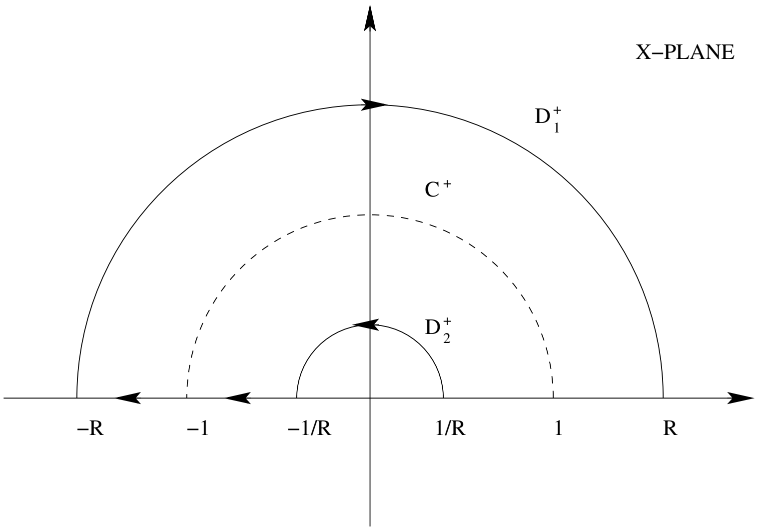

Figure 1: Contours in the upper half plane.

region $x \in [0, 1/R] \cup [R, +\infty]$ the y-integral is gaussian and the integration straightforward: the integral $\int dy A(x) exp {-\frac{A^2(x)}{4}} y^2$ equals (up to a factor $\sqrt{\pi}$) $+1$ for $x \in [R, +\infty]$ and -1 for $x \in [0, 1/R]$. Let us begin with the $a = +1$ term in (3.5). To compute the full integral, one needs to give a prescription to handle the region $[1/R, R]$. We analytically continue the x integral in the complex plane, by choosing a contour as shown in figure 1 ; since $a = +1$ we close the contour in the upper half plane. The arcs $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ are semi-circles of radius R and $\frac{1}{R}$ respectively, while $\mathcal{C}^+$ is the unit semi-circle[4]. We assume positive orientation clockwise. Thus for the x-integration contour we have, schematically:

$$\int_{\frac{1}{R}}^{R} = -\int_{\mathcal{D}_2^+} + \int_{-\frac{1}{R}}^{-1} + \int_{-1}^{-R} + \int_{\mathcal{D}_1^+} , \tag{3.7}$$

where the integrand is that of (3.6), with the y-integration still to be done. At this stage we perform the y gaussian integral, the reason this can be done is that on the deformed contour the gaussian integral is well defined. The integral: $\int_0^\infty \mathrm{d}y A(x) e^{-\frac{A^2(x)}{4} y^2}$, is proportional to $+1(-1)$ for $|x| > 1 (|x| < 1)$, as a result, in (3.7) the first and the second terms on the right-hand side flip sign. Now it is convenient to further split the integral between $-\frac{1}{R}$ and $-R$ into two contributions, using the following relations:

$$\begin{aligned} \int_{-1}^{-R} + \int_{\mathcal{D}_1^+} &= \int_1^R + \int_{\mathcal{C}^+} \\ \int_{-1}^{-\frac{1}{R}} + \int_{\mathcal{D}_2^+} &= -\int_{\frac{1}{R}}^{1} + \int_{\mathcal{C}^+} . \end{aligned} \tag{3.8}$$

After substituting the above deformation of the contours we obtain the contours shown in figure 2. Therefore, using (3.8) we can finally write the equation (3.5), for $a = +1$ as the

[4] We write the superscript + to remind the reader that these contours are in the upper half plane

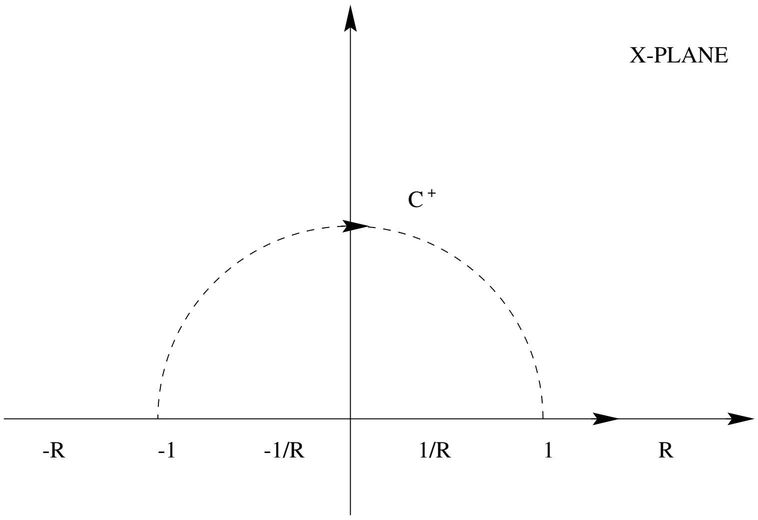

Figure 2: Final contour in the upper half plane.

x-integral:

$$\mathcal{K}\left(\sqrt{z}\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}\left(\frac{1}{x}-1\right)e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)} - 2\sqrt{z}\int_0^\pi \mathrm{d}\theta \sin\frac{\theta}{2}e^{-z\cos\theta}\right). \tag{3.9}$$

Here we have transformed the integral between 0 and 1 to an integral between 1 and ∞ by the change of variable $x \to \frac{1}{x}$ and the second integral is the contribution from the contour $\mathcal{C}^+$, parameterized as $e^{i\theta}$. $\mathcal{K}$ refers to the overall constant that we have not kept track of. Since we are interested in comparing the target space fields of the (m,n) brane to that of the $(1,1)$ brane it is not necessary to keep track of the overall constant.

To (3.9) we should now add the contribution coming from the $a=-1$ term in (3.5). To compute this contribution we follow the same pattern that led to (3.9), the only difference being that, since $a=-1$ and $A(x)=\ln x + i\pi$, the contour integrals close in the lower half plane, as showed in figure 3 Now we introduce $\mathcal{D}_1^-$, $\mathcal{D}_2^-$ and $\mathcal{C}^-$ with radii respectively R, $\frac{1}{R}$ and 1, where the superscript $-$ points out that the contours close in the lower half plane. These semi-circles are positively oriented anti-clockwise. Thus we have

$$\int_{\frac{1}{R}}^{R} = -\int_{\mathcal{D}_2^-} + \int_{-\frac{1}{R}}^{-1} + \int_{-1}^{-R} + \int_{\mathcal{D}_1^-} . \tag{3.10}$$

Again, now the integrals over p and τ can be done, leading us to

$$\begin{aligned} \int_{-1}^{-R} + \int_{\mathcal{D}_1^-} &= \int_1^R + \int_{\mathcal{C}^-} \\ \int_{-1}^{-\frac{1}{R}} + \int_{\mathcal{D}_2^-} &= -\int_{\frac{1}{R}}^{1} + \int_{\mathcal{C}^-} \end{aligned} \tag{3.11}$$

from which we obtain the same result as in (3.9). The final form of the contours are given in figure 4. Thus the result of evaluating the integral (3.5) is basically (3.9).

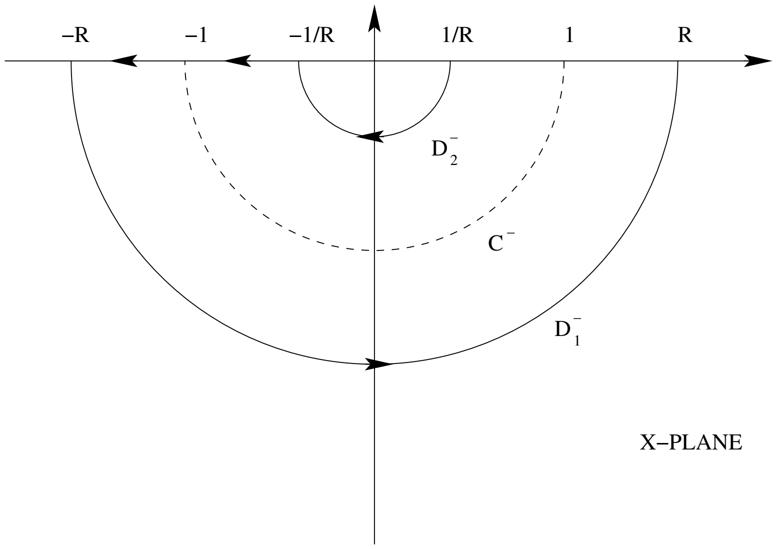

Figure 3: Contours on the lower half plane .

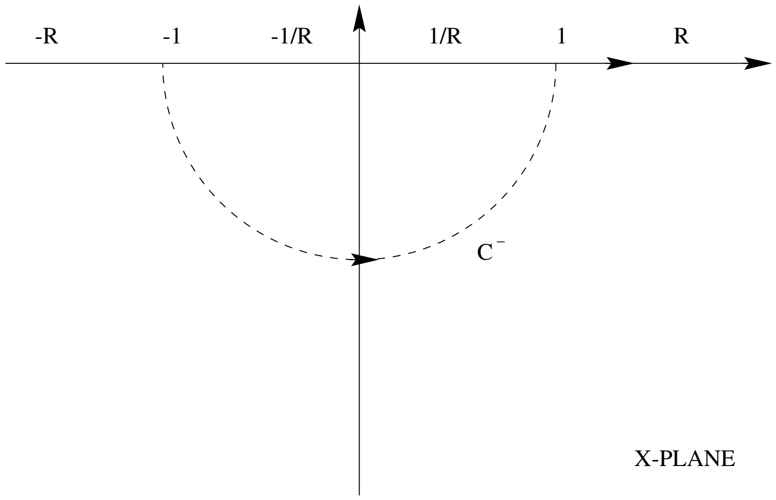

Figure 4: Final contour on the lower half plane.

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The formula (3.9) can in fact be simplified considerably. Using the integral representation of the modified Bessel functions $I_{1/2}$ and $I_{-1/2}$ [16], we have

$$\begin{aligned} I_{1/2}(z) + I_{-1/2}(z) &= -\frac{1}{\pi}\left[\int_0^\infty \frac{dx}{\sqrt{x}}\left(\frac{1}{x}-1\right) - 2\int_0^\pi d\theta e^{z\cos\theta}\sin\frac{\theta}{2}\right], \\ &= -\frac{1}{\sqrt{2\pi z}}e^z \end{aligned} \tag{3.12}$$

Substituting the above formula in (3.9) we see that it is given by $\mathcal{K}e^z$. Thus the RR scalar field strength produced by the (1, 1) ZZ brane is proportional to e^{+z}, which confirms the analysis of [11] that the (1, 1) ZZ–branes are localized at $\phi \to +\infty$. Also note that e^z

satisfies the equation of motion as it satisfies the equation

$$\left(z\frac{\partial}{\partial z}-z\right)e^{z}=0\,. \tag{3.13}$$

Therefore from (2.16) we see that the $(1,1)$ D-instanton distributed uniformly in the Euclidean time direction sources the magnetic RR scalar.

In fact one can directly check that equation (3.9) satisfies the equations of motion, this is useful to generalize to the case of (m,n) brane. To do this one should apply the second order differential operator (2.2) directly to the potential (3.3). This is equivalent to apply the first order differential operator $z\frac{\partial}{\partial z}-z$ directly to the field strength given by (3.9). To check this, the following formula has to be used:

$$\begin{aligned}\left(z\frac{\partial}{\partial z}-z\right)\sqrt{z}\int_1^\infty\frac{\mathrm{d}x}{\sqrt{x}}\left(\frac{1}{x}-1\right)e^{-\frac{z}{2}(x+\frac{1}{x})} &= -\sqrt{z}\int_1^\infty \mathrm{d}x\frac{\partial}{\partial x}\left(\sqrt{x}\left(\frac{1}{x}+1\right)e^{-\frac{z}{2}(x+\frac{1}{x})}\right)\\ &= 2\sqrt{z}e^{-z}\,.\end{aligned} \tag{3.14}$$

Note that, due to the formula above, for the equation of motion to be satisfied, only the integration limits matter. What happens is that after applying $z\frac{\partial}{\partial z}-z$ the integrand becomes a total derivative in x and the boundary contributions from the two terms in (3.9), from $x=1$ and $\theta=0$, cancel each other. The fact that $z\frac{\partial}{\partial z}-z$ annihilates the field strength, implies that the latter is proportional to e^z. This coincides with the non-normalizable solution of the equations of motion discussed in [8] and reviewed in section 2.

3.1 (m,n) branes : the RR field

In this subsection we will extend the previous discussion to the case $(m,n)\neq(1,1)$. Let us take the mini-superspace limit of the RR wave functions (2.10). We have to distinguish the cases of n even and n odd: in the case of n even, one sees from (2.10) that $\Psi_{RR}(p;m,n)$ gives a factor npb for b small and this results in an extra power of p in the numerator, compared to the previous case, in the integrand for the field strength. By regularizing the integral as before, one can verify that in this case the second integration (in τ) gives a vanishing result. In the case of n odd $\Psi_{RR}(p;m,n)$ becomes, up to a phase, the same as $\Psi_{RR}(p;m,1)$ in the $b\to0$ limit. The resulting integral will be evaluated along the same lines indicated in the previous subsection for the (1,1) case. We have in turn to consider separately the cases of m even and m odd. In the first case, due to the reflection properties of $\Psi_{p\pm}$ and $\Psi_{RR}(p;m,n)$ discussed in section 2, we see that only Ψ_{p-} can enter in the integral of the type 3.2. Therefore we will need to evaluate:

$$i^{m+n+1}\int \mathrm{d}p\,\sqrt{z}\left(K_{\frac{1}{2}+ip}(z)-K_{\frac{1}{2}-ip}(z)\right)\frac{1}{p^2}\sinh[\pi mp],\qquad m\text{ even}\,. \tag{3.15}$$

For the same reasons, for m odd only $\Psi_{p+}(z)$ can contribute and we have the integral:

$$i^{m+n}\int \mathrm{d}p\,\sqrt{z}\left(K_{\frac{1}{2}+ip}(z)+K_{\frac{1}{2}-ip}(z)\right)\frac{1}{p^2}\cosh[\pi mp],\qquad m\text{ odd}. \tag{3.16}$$

Notice that again the Γ functions and the p-dependent phases cancel between $\Psi_{RR}(p;m,n)$ and the normalization of $(\Psi_{p\pm}(z))^*$.

Let us begin with the latter case, m odd: we apply to (3.16) the operator (2.1) to obtain the field strength. We need then to evaluate integrals of the type:

$$\int_{-\infty}^{\infty} \mathrm{d}p \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}} \int_0^\infty \mathrm{d}\tau\, p \left(\frac{1}{x} - 1\right) e^{ip\ln x} e^{\pi mpa} e^{-\tau p^2} e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)}, \tag{3.17}$$

where the factor containing $a = \pm 1$ arises when writing $\cosh \pi mp$ in terms of exponentials. We are facing the same convergence problems we had in the case $m = 1$ and we follow the same procedure. Let us start with $a = +1$. To compute this integral, we choose again to promote x to a complex variable and deform the contour of integration after having regulated the integral; but this time, due to the integer $m > 1$, the divergence in the y-integral is worse. To cure this problem, it is sufficient to continue the semi-circles $\mathcal{D}$ and $\mathcal{C}$ to a contour ending into the $\frac{m+1}{2}$-th sheet in the (infinite) covering of the x plane . In other words, we take a "spiral" that winds around the origin counterclockwise(clockwise) with an angle of $m\pi$, starting from a given point on the real x-axis when a is positive(negative). Then, the previous interval $(Re^{i\pi}, \frac{1}{R}e^{i\pi}))$ on the negative real x axis, is replaced now by $(Re^{im\pi}, \frac{1}{R}e^{im\pi}))$, with some different R. This is exactly the angle needed to undo the term $-im\pi$ in $\ln x - im\pi$ in order to compute the gaussian y-integral. We then proceed with the x-contour deformation argument following the same steps outlined for the (1,1)-brane, arriving at:

$$\sqrt{z} \int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}} \left(\frac{1}{x} - 1\right) e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)} - 2\sqrt{z} \int_0^{m\pi} \mathrm{d}\theta \sin\frac{\theta}{2} e^{-z\cos\theta} \tag{3.18}$$

The same procedure is utilized when $a = -1$, leading again to (3.18). Moreover, one can check, by using formula (3.14), that the equations of motions are satisfied. The reason is that, due to (3.14), only the boundary terms matter and in (3.18) the boundary term is 1 in the first integral and in 0 in the second cancel each other, while the others vanish.

Let us look more closely at (3.18): The m dependence is rather peculiar; in fact the second integral obeys the following property

$$\begin{aligned}\int_0^\pi \mathrm{d}\theta \sin\frac{\theta}{2} e^{-z\cos\theta} &= \int_\pi^{2\pi} \mathrm{d}\theta \sin\frac{\theta}{2} e^{-z\cos\theta} \\ &= -\int_{2\pi}^{3\pi} \mathrm{d}\theta \sin\frac{\theta}{2} e^{-z\cos\theta} = -\int_{3\pi}^{4\pi} \mathrm{d}\theta \sin\frac{\theta}{2} e^{-z\cos\theta}\end{aligned} \tag{3.19}$$

that can be proven by using the properties of the trigonometric functions and the fact that the integrand is a periodic function with period 4π. This means that, in (3.18), where m is odd, if we split the second integral in m integrals over intervals of length π, all the terms pairwise cancel except one, leading us back to the result (3.9). It seems that, at least in the mini-superspace approximation, (m, n)-branes with m and n odd, behave exactly as (1,1) branes with respect to the RR field, i.e. they have the same RR charge. In fact, just as for the case of the $(1, 1)$ brane the result of the integral in (3.18) is just proportional to e^z. Therefore it is clear that the RR field sourced by the (m, n) brane satisfies the equation of motion (3.13).

Let us suppose now that m is an even integer. The two contributions arising from (3.15), when computing the field strength are of the form:

$$i^{m+n}\int_{-\infty}^{\infty}\mathrm{d}p\int_0^{\infty}\frac{\mathrm{d}x}{\sqrt{x}}\int_0^{\infty}\mathrm{d}\tau\, p\,(\frac{1}{x}+1)e^{ip\ln x}e^{\pi mpa}e^{-\tau p^2}e^{-\frac{z}{2}(x+\frac{1}{x})}. \tag{3.20}$$

The differences with respect to the m-odd case, are the sign in the factor $1+\frac{1}{x}$ and the fact that now the contributions with $a=\pm$ carry a different relative sign that comes directly from $\sin m\pi p$. The integral can be computed using the same procedure which has been detailed for the case of m odd. Again, after doing the p and τ integrals, one finds three contributions:

$$\sqrt{z}\left(-\int_0^1+2\int_{\mathcal{C}}+\int_1^{\infty}\right)\frac{\mathrm{d}x}{\sqrt{x}}\left(\frac{1}{x}+1\right)\,e^{-\frac{z}{2}(x+\frac{1}{x})}. \tag{3.21}$$

This time, due to the factor $\frac{1}{x}+1$ the first and the last integral cancel each other. What remains is:

$$2i\sqrt{z}\int_0^{m\pi}\mathrm{d}\theta\cos\frac{\theta}{2}e^{-z\cos\theta}. \tag{3.22}$$

But now one can show that this integral vanishes. In fact, due to the periodicity of the integrand, the relevant integral to compute is just the one from 0 to 2π. But this integral vanishes since

$$\begin{aligned}\int_0^{\pi}\mathrm{d}\theta\cos\frac{\theta}{2}e^{-z\cos\theta} &= \int_{\pi}^{2\pi}\mathrm{d}\theta\sin\frac{\theta}{2}e^{z\cos\theta}\\ &= -\int_{\pi}^{2\pi}\mathrm{d}\theta'\cos\frac{\theta'}{2}e^{-z\cos\theta'}\end{aligned} \tag{3.23}$$

where $\theta'=-\theta+3\pi$. The same computation carries on for the case of $a=-1$ with exactly the same vanishing result. Combining all the cases, we find that that (m,n)-branes are decoupled from the RR case except when both m and n are odd. For this case, the RR charge is identical to that of the $(1,1)$ brane.

3.2 (m,n) branes: the NS-NS field

We proceed now to discuss the NSNS field produced by (m,n) ZZ branes, which, in particular, contains the information about their tension. We have the wave-function corresponding to the NSNS ground state (2.5) in the semi-classical limit and the exact disc NSNS one-point function, (2.9). In the $b\to 0$ limit the leading term in the latter is of order b and proportional to n. More precisely, after taking into account of a $1/b$ factor from the p-measure, equation (3.2) is proportional to

$$n\int\mathrm{d}p\,K_{ip}(z)\frac{1}{p^2}\sinh(\pi mp)p\,, \tag{3.24}$$

where the factor np comes from the $b\to 0$ limit of (2.9). Here the p appearing in (3.24) is the p of (2.9) divided by b. Note that the only n dependence is a multiplicative factor outside the integral. The integral (3.24) can be reduced to integrals of the kind

$$n\int\mathrm{d}p\int_0^{\infty}\mathrm{d}t\int_0^{\infty}\frac{\mathrm{d}x}{x}e^{ip\log x}e^{\pi mpa}pe^{-tp^2}e^{-\frac{z}{2}(x+\frac{1}{x})}, \tag{3.25}$$

where a can be ± 1 and the exponential containing a arises from the $\sinh(\pi m p)$. We can evaluate this integral by following the method discussed earlier. We arrive at:

$$-ni\int_0^1 \frac{\mathrm{d}x}{x}\, e^{-\frac{z}{2}(x+\frac{1}{x})} + ni\int_1^\infty \frac{\mathrm{d}x}{x}\, e^{-\frac{z}{2}(x+\frac{1}{x})} + 2ni\int_{\mathcal{C}} \frac{\mathrm{d}x}{x}\, e^{-\frac{z}{2}(x+\frac{1}{x})}. \tag{3.26}$$

The first two terms cancel each other due to the symmetry properties of the integrand under the transformation $x \to \frac{1}{x}$; the last term is integrated along the contour $\mathcal{C}$ that winds around the origin covering an angle of $m\pi$ and can thus be written as:

$$2mn\int_0^{\pi m} \mathrm{d}\vartheta e^{-z\cos\vartheta} = \mathcal{K} mn I_0(z)\,, \tag{3.27}$$

where $I_0(z)$ is the (modified) Bessel function [16] and $\mathcal{K}$ refers to the normalization constant which we have not kept track of. Note that $I_0(z)$ is annihilated by the operator $(-(z\frac{\partial}{\partial z})^2 + z^2)$, which is the mini-superspace hamiltonian (for zero energy) in the NS sector, i.e. equation (2.6) for $p = 0$ and without the linear term in z. In fact I_0 is the exponentially growing solution of this differential equation, $I_0 \sim e^z/\sqrt{z}$ for large z. This result is in agreement with what was found for the RR case, where also the field was exponentially growing at large z. Finally, (3.27) indicates that the tension of (m, n) branes, at least in the mini-superspace approximation is proportional to the product mn.

4. Beyond the semi-classical approximation

We have already stressed that for the case of non-critical type 0B theory we are interested in, the Liouville dynamics is far from the semi-classical regime of small b (or large b as $b \to 1/b$ is a symmetry) since b is actually at the self-dual point $b = 1$. What is missing, in order to extend the strategy of the previous section beyond the semi-classical limit, are the expressions for the ground state wave-functions. In general we expect the equation of motions to receive higher order corrections in b. Unfortunately, the generalizations of the mini-superspace wave functions (2.3) and (2.5) are presently not known. We have however some non-perturbative information about the exact wave-functions. This is given by the exact reflection coefficients both for RR and NSNS ground states. In fact, they can be read-off from the expressions for the boundary states wave-functions, (2.9) and (2.10). Setting $b = 1$, for the NSNS sector these are given by:

$$S^{NS}(p) = -\left(\frac{\mu}{2}\right)^{-2ip} \frac{\Gamma^2(ip)}{\Gamma^2(-ip)}\,, \tag{4.1}$$

and for the RR sector they are given by:

$$S^{RR}(p) = \left(\frac{\mu}{2}\right)^{-2ip} \frac{\Gamma^2(\frac{1}{2}+ip)}{\Gamma^2(\frac{1}{2}-ip)}\,. \tag{4.2}$$

Here again we have denoted with μ what actually is the renormalized cosmological constant defined in (2.11). The exact, quantum wave functions obey reflection relations of the type (2.7) with $S^{NS,RR}$ given in (4.1) and (4.2). The semi-classical wave-functions (2.5) behave, in $z \to 0$ limit, far away from the Liouville potential, as a superposition of an incoming and an outgoing wave with a relative reflection coefficient. We assume that this feature still holds beyond the mini-superspace approximation, with the mini-superspace reflection coefficients replaced by the quantum version, (4.1) and (4.2). We then postulate the existence of "quantum" wave functions, expressible in the form:

$$\Psi^{NS}(p,z) = -\frac{(\mu/2)^{-ip}}{\Gamma^2(-ip)} \int_0^\infty \mathrm{d}x\, x^{ip-1} f^{NS}(x,z) \tag{4.3}$$

$$\Psi^{RR}_{\pm}(p,z) = \frac{(\mu/2)^{-ip}}{\Gamma^2(\frac{1}{2}-ip)} \int_0^\infty \mathrm{d}x\, x^{ip-1} f^{RR}_{\pm}(x,z)\,, \tag{4.4}$$

where the functions $f^{NS,RR}(x,z)$ obey the relations:

$$\begin{aligned} f^{NS}(x,z) &= f^{NS}\left(\frac{1}{x},z\right) \\ f^{RR}_{\pm}(x,z) &= \pm f^{RR}_{\pm}\left(\frac{1}{x},z\right) \end{aligned} \tag{4.5}$$

that is similar to the properties of (2.3) and (2.5), that can be obtained from the integral representation of the modified Bessel function (3.4).

The wave functions (4.3) and (4.4) should reduce in the weakly coupled region (i.e. in the $z \to 0$) limit to a superposition of incoming and outgoing waves, the relative phase being given by the reflection coefficients (4.1) and (4.2), respectively. These can be interpreted as conditions on the wave-functions $f^{NS,RR}(x,z)$. From these conditions one finds the $z \to 0$ behavior of the functions $f^{NS,RR}(x,z)$

$$f^{NS}(x,z) \to_{z\to 0} \left(K_0\left(\sqrt{2xz}\right) - K_0\left(\sqrt{\frac{2z}{x}}\right)\right) \tag{4.6}$$

$$f^{RR}_{+}(x,z) \to_{z\to 0} \sqrt{z}\left(\sqrt{x}K_0\left(\sqrt{2xz}\right) + \frac{1}{\sqrt{x}}K_0\left(\sqrt{\frac{2z}{x}}\right)\right) \tag{4.7}$$

$$f^{RR}_{-}(x,z) \to_{z\to 0} \sqrt{z}\left(-\sqrt{x}K_0\left(\sqrt{2xz}\right) + \frac{1}{\sqrt{x}}K_0\left(\sqrt{\frac{2z}{x}}\right)\right). \tag{4.8}$$

$$\tag{4.9}$$

The fact that these expression reproduce the right asymptotic behaviour can be explicitly verified. This is done by performing the x-integral in (4.3) and (4.4) with the use of the integral representation for the modified Bessel function given in (3.4)[5].

We can push further the analogy with the mini-superspace wave functions (2.3), by demanding the analog of equations (2.1) and (2.2). Namely, we postulate the existence of differential operators $D^2_{NS}(z,\frac{\partial}{\partial z})$ and $D^{\pm}_{RR}(z,\frac{\partial}{\partial z})$ acting on the wave-functions such that:

$$\begin{aligned} \mathcal{D}^{+}_{RR}\Psi^{RR}_{+}(z,p) &= p\Psi^{RR}_{-}(z,p) \\ \mathcal{D}^{-}_{RR}\Psi^{RR}_{-}(z,p) &= p\Psi^{RR}_{+}(z,p) \end{aligned} \tag{4.10}$$

[5] One can easily generalize (4.3), (4.4) and (4.6) to the case of $b \neq 1$. The subsequent analysis however becomes more involved.

and

$$\mathcal{D}^2_{RR}\Psi^{RR}_{\pm}(z,p) \equiv \mathcal{D}^+_{RR}\mathcal{D}^-_{RR}\Psi^{RR}_{\pm}(z,p) = 0\,, \tag{4.11}$$

as well as:

$$\mathcal{D}^2_{NS}\Psi^{NS}(z,x) = 0\,. \tag{4.12}$$

More generally, we may assume that there is a b-dependent family of each of these operators, interpolating between the mini-superspace expressions (2.1) and (2.2) in the $b \to 0$ limit and some (unknown) expression for the case $b = 1$. On the other hand, the factorization relation, (4.11), is a consequence of world-sheet supersymmetry, which is an exact symmetry, independent of b. Notice that it is not obvious that the differential operators $D^2_{NS}(z, \frac{\partial}{\partial z})$ and $D^{\pm}_{RR}(z, \frac{\partial}{\partial z})$, remain of order two and one respectively. Nevertheless, we may expect that the the structure of equations (4.10) and (4.12) is still valid, since this arises from supersymmetry and from the fact the t-dependent part of the full wave-function is in any case a plane-wave, e^{ipt}. Regarding (4.3) and (4.4), the assumption that the wave-functions can be written in that way seems rather natural, being just a definition of their Fourier transforms. The property (4.5), on the other hand, is a symmetry property obeyed by the corresponding mini-superspace wave functions which we have assumed to be true beyond the $b \to 0$ limit. From (4.4) and (4.10) it follows, by a partial integration, that $f^{RR}_{\pm}(x,z)$ satisfies the following identity:

$$\mathcal{D}^{\pm}_{RR}(z) f^{RR}_{\pm}(x,z) = ix\frac{\partial}{\partial x} f^{RR}_{\mp}(x,z)\,. \tag{4.13}$$

This equation generalizes the relation (3.14) beyond the mini-superspace limit, and will be used when we will check the equation of motion for the RR field strength.

4.1 The RR field

Let us begin with the evaluation of the RR field strength. Again, the idea is to write down an integral of the kind of (3.3) with the full one-point function and the wave functions (4.3) and (4.4) (with the appropriate parity, depending on m and n being even or odd). Generically we have to evaluate:

$$\int \mathrm{d}p\,\mathrm{d}x\,\frac{1}{p^2} x^{ip-1}\left(f^{RR}_{\pm}(x,z)\right)^* \sin\left[\pi m\left(\frac{1}{2}+ip\right)\right] \sin\left[\pi n\left(\frac{1}{2}+ip\right)\right]. \tag{4.14}$$

We are interested in computing the field strength out of these integrals: thus we apply the operators $\mathcal{D}^{\pm}_{RR}$ (4.10) to obtain an additional factor of p. Moreover the operator (4.10) changes the parity of the function $f^{RR}_{\pm}(x,z)$, i.e. flips $f^{RR}_{\pm}(x,z)$, to $f^{RR}_{\mp}(x,z)$. We therefore need to consider, distinguishing explicitly the two parities:

$$\int \mathrm{d}p\,\mathrm{d}x\,\frac{p}{p^2} x^{ip-1} f^{RR}_{-}(x,z) \begin{cases} \sinh(\pi m p)\sinh(\pi n p) & \text{if } m = \text{even, } n = \text{even} \\ \cosh(\pi m p)\cosh(\pi n p) & \text{if } m = \text{odd, } n = \text{odd} \end{cases} \tag{4.15}$$

and

$$\int \mathrm{d}p\,\mathrm{d}x\,\frac{p}{p^2} x^{ip-1} f^{RR}_{+}(x,z) \begin{cases} \sinh(\pi m p)\cosh(\pi n p) & \text{if } m = \text{even, } n = \text{odd} \\ \cosh(\pi m p)\sinh(\pi n p) & \text{if } m = \text{odd, } n = \text{even}\,. \end{cases} \tag{4.16}$$

Here the gamma-function squared in the numerator, as well as the p-dependent phase, cancel with the normalization of the wave-function $\Psi_{\pm}^{RR}$: thus the integral in p is a (divergent) gaussian integral that need to be regulated and can be computed as we did in the mini-superspace approximation. Indeed, the p and τ integrals are basically the same as before: the factor $e^{\pi p a}$, in the notation of section 3, which before had $a = \pm 1$, now has generically $a = \pm m \pm n$. Using the contour deformation procedure already explained, one finds for the final x-integrals the expressions, up to m, n dependent phases and overall constants:

$$\begin{aligned}
&\left(2\int_{e^{i\pi(m+n)}}^{1} -2\int_{e^{i\pi(m-n)}}^{1}\right)\frac{1}{x}f_{-}^{RR}(x,z). \quad \text{if } m=\text{even},\ n=\text{even}\\
&\left(2\int_{1}^{\infty} +2\int_{e^{i\pi(m+n)}}^{1} +2\int_{e^{i\pi(m-n)}}^{1}\right)\frac{1}{x}f_{-}^{RR}(x,z). \quad \text{if } m=\text{odd},\ n=\text{odd}\,.
\end{aligned} \tag{4.17}$$

The remaining two cases are:

$$\begin{aligned}
&\left(2\int_{e^{i\pi(m+n)}}^{1} +2\int_{e^{i\pi(m-n)}}^{1}\right)\frac{1}{x}f_{+}^{RR}(x,z) \quad \text{if } m=\text{even},\ n=\text{odd}\\
&\left(2\int_{e^{i\pi(m+n)}}^{1} -2\int_{e^{-i\pi(m-n)}}^{1}\right)\frac{1}{x}f_{+}^{RR}(x,z) \quad \text{if } m=\text{odd},\ n=\text{even}\,.
\end{aligned} \tag{4.18}$$

In both cases we have used the properties:

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{x}\, f_{-}^{RR}(x,z) &= -\int_1^{\infty}\frac{\mathrm{d}x}{x}\, f_{-}^{RR}(x,z)\\
\int_0^1 \frac{\mathrm{d}x}{x}\, f_{+}^{RR}(x,z) &= \int_1^{\infty}\frac{\mathrm{d}x}{x}\, f_{+}^{RR}(x,z)
\end{aligned} \tag{4.19}$$

that follow from (4.5). Using (4.13) and the property obeyed by $f_{\pm}^{RR}(x,z)$ given above, one can verify that the integrals (4.17) and (4.18) are annihilated, in the $z \to 0$ limit, by the operators $\mathcal{D}_{RR}^{-}$ and $\mathcal{D}_{RR}^{+}$ respectively, i.e. that the equations of motion for the field strength are satisfied. More generally, for arbitrary z, requiring the equations of motion to be satisfied, will just put appropriate conditions for the functions $f_{\pm}^{RR}(x,z)$ at the end points of the integration contours in (4.17) and (4.18).

One can also evaluate (4.17) and (4.18) in the limit $z \to 0$ using the asymptotic expression for $K_0(z) \sim -lnz$ [16], at least for the integrals which involve a finite range of integration for x. Let us start from the case (m,n) (even,even): Using the explicit form of f_{-}^{RR} given above, one finds that the relevant integral in (4.17) becomes proportional to:

$$\left(\int_0^{\pi(m+n)} - \int_0^{\pi(m-n)}\right)\mathrm{d}\theta\left(\ln z\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right), \tag{4.20}$$

By doing the explicit θ-integral one finds:

$$\left[-2\ln z\left((-)^{\frac{m+n}{2}}-1\right)+4\left((-)^{\frac{m+n}{2}}-1\right)\right]-[n\to -n]\,, \tag{4.21}$$

which vanishes for (m,n) (even,even). For the cases (m,n) (even,odd), using the asymptotic expression for f_{+}^{RR} given in (4.6), one obtains:

$$\left(\int_0^{\pi(m+n)} + \int_0^{\pi(m-n)}\right)\mathrm{d}\theta\left(\ln z\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right). \tag{4.22}$$

Performing the θ-integration as before, one finds:

$$\left[-2\ln z(-)^{\frac{m+n+1}{2}}+4(-)^{\frac{m+n+1}{2}}\right]+[n\to -n]\,, \tag{4.23}$$

which vanishes. Similarly for (m,n) (odd,even) we have:

$$\left(\int_0^{\pi(m+n)}-\int_0^{\pi(m-n)}\right)\mathrm{d}\theta\left(\ln z\cos\frac{\theta}{2}+\theta\sin\frac{\theta}{2}\right), \tag{4.24}$$

which gives:

$$\left[-2\ln z(-)^{\frac{m+n+1}{2}}+4(-)^{\frac{m+n+1}{2}}\right]-[n\to -n]\,, \tag{4.25}$$

which again vanishes. Finally, we are left with the case (m,n) (odd,odd). The corresponding expression contains an integral which is independent of m,n, plus two integrals that may depend on (m,n) and can be evaluated, in the $z\to 0$ limit, as in the cases before. By an explicit calculation, one can indeed verify that, the (m,n) dependence drops out. The results we have found for the RR field, starting from the "exact"wave-function in the $z\to 0$ limit, agree with the mini-superspace analysis of section 3.

4.2 The NS-NS field

Let us now consider the NS case. The NS-NS field is proportional to

$$\int \mathrm{d}p\,\mathrm{d}x\,\frac{1}{p^2}x^{ip-1}\left(f^{NS}(x,z)\right)^*p\sinh(\pi mp)\sinh(\pi np)\Gamma^2(ip)\left(\frac{\mu}{2}\right)^{-ip}, \tag{4.26}$$

where we have used $x\Gamma(x)=\Gamma(x+1)$. The factor involving the gamma-function above, as well as a p- dependent phase cancels, as usual, with the normalization of f^{NS}. As a result, the integrand is an even function of p, after taking into account the inversion property (4.5) of f^{NS}. Proceeding as it was repeatedly done before, we end up with an expression proportional to:

$$\left(\int_0^{\pi(m+n)}-\int_0^{\pi(m-n)}\right)\mathrm{d}\theta\left(K_0\left(\sqrt{2zx}\right)-K_0\left(\sqrt{\frac{2z}{x}}\right)\right),$$

where $x=e^{i\theta}$. Using again the behaviour $K_0(z)\sim -\ln z$ for $z\to 0$, one finds that the expression (4.27) becomes

$$\left(\int_0^{\pi(m+n)}-\int_0^{\pi(m-n)}\right)\mathrm{d}\theta\theta \tag{4.27}$$

which is proportional to mn. This is again in agreement with the result of section 3. There the NS-NS tachyon field source by the (m,n) brane is given by (3.27), for $z\to 0$ limit $I_0(z)\to 1$ therefore, this reduces to what we have found above.

5. Probing ZZ with FZZT

In this section we provide a further check on the results, obtained in the previous sections, about the field configurations produced by ZZ branes by using FZZT branes [9, 10] as probes. The idea is thus to look at a mixed, ZZ and FZZT annulus partition function, which describes the exchange of closed string states between ZZ and FZZT branes, and extract from this information about the NSNS or RR field produced by the ZZ brane. The annulus partition function with mixed ZZ–FZZT boundary conditions for either the NSNS or RR closed string sectors, $\mathcal{Z}^{NS,R}(m,n|\sigma)$, depends on the ZZ labels (m,n), whose meaning has been explained in section 2, and on a continuous parameter, σ, for FZZT (Neumann) boundary conditions. This latter indeed characterizes continuous, non-degenerate representations of Liouville theory and is related to the boundary cosmological constant μ_B through $\mu_B = \cosh \pi b \sigma$, for arbitrary b.

5.1 The NS-NS field

Given $\mathcal{Z}(m,n|\sigma)$ in the NSNS sector, following [11], we introduce the field $\Psi(z)$, encoding information on the target space NSNS background, through:

$$\mathcal{Z}^{NS}(m,n|\sigma) = \int_0^\infty \frac{\mathrm{d}z}{z}\, e^{-\frac{z}{2}\left(x+\frac{1}{x}\right)} \Psi(z), \tag{5.1}$$

where we have introduced the variable $x = e^{\pi\sigma}$ and set $b = 1$. Notice that, in this case $x + 1/x = \mu_B$, and the definition (5.1) is partly motivated by the the intuition from the mini-superspace approximation, where one can see that μ_B is conjugate to z. Indeed, in this approximation, the wave function corresponding to a FZZT brane vanishes roughly for z larger than μ_B^{-1}. To evaluate (5.1) we will compute $\mathcal{Z}^{NS}(\sigma'|\sigma)$ i.e. the annulus amplitude corresponding to FZZT-FZZT boundary conditions, and then analytically continue in the parameter σ, by using the known relation between FZZT and ZZ boundary states, which amounts to the identity [5]:

$$\mathcal{Z}^{NS}(m,n|\sigma) = \mathcal{Z}^{NS}\left(\sigma'_{m,-n} = i(m-n)|\sigma\right) - \mathcal{Z}^{NS}\left(\sigma'_{m,n} = i(m+n)|\sigma\right). \tag{5.2}$$

Similar relations hold in the RR sector.

Let us begin by evaluating the FZZT-FZZT partition function. One can think of it, for example, as expressing the interaction between a D-instanton and a (unstable and space-like) D0-brane in 0B string theory, where for both it is assumed a uniform distribution in the time direction, as in section 3. Notice that the oscillator modes cancel between matter and (super-)ghosts. Therefore, it is given just in terms of the FZZT the boundary state wave functions $\Psi^{NS}(p,\sigma)$ as:

$$\mathcal{Z}^{NS}(\sigma'|\sigma) = \int \mathrm{d}p (\Psi^{NS}(p,\sigma')^* \frac{1}{p^2} \Psi^{NS}(p,\sigma), \tag{5.3}$$

where, up to an overall constant:

$$\Psi^{NS}(p,\sigma) = i\cos(\pi\sigma p) \frac{\Gamma^2(1+ip)}{p} \left(\frac{\mu}{2}\right)^{-ip}. \tag{5.4}$$

Using this, we arrive at:

$$\mathcal{Z}^{NS}(\sigma'|\sigma) = \int \mathrm{d}p \, \frac{\cos(p\pi\sigma')\cos(p\pi\sigma)}{\sinh^2(\pi p)} \frac{1}{p^2}. \tag{5.5}$$

To continue with the evaluation, we find it convenient to act with the operator $x\partial_x = \frac{1}{\pi}\partial_\sigma$ on the identity (5.1) to lower the IR divergence in the momentum integration. The relevant integral is now:

$$\frac{1}{\pi}\frac{\partial}{\partial\sigma}\mathcal{Z}^{NS}(\sigma'|\sigma) = -\int \mathrm{d}p \, \frac{\cos(p\pi\sigma')\sin(p\pi\sigma)}{\sinh^2(\pi p)} \frac{p}{p^2}, \tag{5.6}$$

which, in turn, equals the following integral:

$$\frac{1}{\pi}\frac{\partial}{\partial\sigma}\mathcal{Z}^{NS}(\sigma'|\sigma) = -\frac{1}{2i}\int \mathrm{d}p \, \frac{e^{ip\pi(\sigma+\sigma')} + e^{ip\pi(\sigma-\sigma')}}{\sinh^2(\pi p)} \frac{p}{p^2}. \tag{5.7}$$

We will compute this integral by using the residue theorem; to do this, we will think of p as a complex variable and evaluate the poles of the integrand. We will assume both $\sigma+\sigma'$ and $\sigma-\sigma'$ positive and choose a contour that closes in the upper half plane. At the end of the computation we will show our results to be analytic in σ and σ'. Particular attention should be payed when evaluating the double pole at $p=0$ since it lies on the integration contour: a defining prescription is needed. The contribution from the poles gives (up to an overall coefficient):

$$\begin{aligned}\frac{1}{\pi}\frac{\partial}{\partial\sigma}\mathcal{Z}^{NS}(\sigma'|\sigma) = {} & \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{i\pi(\sigma+\sigma')e^{-k\pi(\sigma+\sigma')} + i\pi(\sigma-\sigma')e^{-k\pi(\sigma-\sigma')}}{ik} + \\ & + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{e^{-k\pi(\sigma+\sigma')} + e^{-k\pi(\sigma-\sigma')}}{k^2} - \\ & - \frac{1}{4}(\sigma+\sigma')^2 - \frac{1}{4}(\sigma-\sigma')^2 - \frac{1}{3}.\end{aligned} \tag{5.8}$$

The structure of the above equation is as follows: the first two lines are arising from the second order poles at $p = ik$, k being a positive integer. The last line is due to the pole at $p=0$ for which a particular prescription is needed. We chose to use the following ϵ–prescription:

$$\frac{1}{\pi}\frac{\partial}{\partial\sigma}\mathcal{Z}^{NS}(\sigma'|\sigma) = -\lim_{\epsilon\to 0}\frac{\pi}{2i}\int \mathrm{d}p \, \frac{e^{ip\pi(\sigma+\sigma')} + e^{ip\pi(\sigma-\sigma')}}{\sinh(\pi(p+i\epsilon))\sinh(\pi(p-i\epsilon))}\frac{p}{(p+i\epsilon)(p-i\epsilon)}. \tag{5.9}$$

Let us focus only on the first exponential in the above equation. The evaluation of the $p = i\epsilon$ pole gives:

$$O\left(\frac{1}{\epsilon^2}\right) + O\left(\frac{1}{\epsilon}\right) - \frac{1}{4}\left(\sigma+\sigma'\right)^2 - \frac{1}{6} + O(\epsilon) + \cdots. \tag{5.10}$$

Note that, as in [11] the the pole at $p - i\epsilon$ contains divergent pieces from which the finite part as $\epsilon \to 0$ has to be extracted. This leads us to the final result of (5.8). Now we proceed in summing the series, by relying on the formulas

$$\sum_{k=1}^{\infty}\frac{x^k}{k} = -\log(1-x), \qquad \sum_{k=1}^{\infty}\frac{x^k}{k^2} = \mathrm{Li}_2(x), \tag{5.11}$$

where x is understood to lie in the convergence domain of the respective sum. These sums allow us to rewrite (5.8) as:

$$\begin{aligned}\frac{1}{\pi}\frac{\partial}{\partial\sigma}\mathcal{Z}^{NS}(\sigma'|\sigma) =& -\frac{1}{\pi}(\sigma+\sigma')\log\left(1-e^{-\pi(\sigma+\sigma')}\right)-\frac{1}{\pi}(\sigma-\sigma')\log\left(1-e^{-\pi(\sigma-\sigma')}\right)+\\ &+\frac{1}{\pi^2}\mathrm{Li}_2\left(e^{-\pi(\sigma+\sigma')}\right)+\frac{1}{\pi^2}\mathrm{Li}_2\left(e^{-\pi(\sigma-\sigma')}\right)-\\ &-\frac{1}{4}(\sigma+\sigma')^2-\frac{1}{4}(\sigma-\sigma')^2-\frac{1}{3}.\end{aligned} \tag{5.12}$$

This result holds for $\sigma+\sigma'>0$ and $\sigma-\sigma'>0$.

We now extend it to all values of σ an σ' by analytic continuation. For this purpose let us prove that eq. (5.8) is analytic in $\sigma+\sigma'$ and $\sigma-\sigma'$. Let us start with $\sigma+\sigma'$. Writing σ for $\sigma+\sigma'$, our result is of the form:

$$-\frac{\sigma}{\pi}\log(1-e^{-\pi\sigma})+\frac{1}{\pi^2}\mathrm{Li}_2(e^{-\pi\sigma})-\frac{1}{4}\sigma^2-\frac{1}{6}, \tag{5.13}$$

for $\sigma>0$. On the other hand, for $\sigma<0$ we have:

$$-\frac{\sigma}{\pi}\log(1-e^{\pi\sigma})-\frac{1}{\pi^2}\mathrm{Li}_2(e^{\pi\sigma})+\frac{1}{4}\sigma^2+\frac{1}{6}, \tag{5.14}$$

where an overall minus sign is due to the change of orientation in the contour of integration, when picking up poles in the lower half plane. Analyticity means that the two expressions are actually the same function defined in the complex σ plane. In other words, if we continue the first expression to $\sigma<0$ we get the second expression and vice-versa if we continue the second expression to $\sigma>0$. This is easily proven by using the following property of the di-logarithm function [17]:

$$\mathrm{Li}_2(y)+\mathrm{Li}_2\left(\frac{1}{y}\right)=\frac{\pi^2}{3}-\frac{1}{2}\log^2 y-i\pi\log y, \tag{5.15}$$

that holds for $y>1$. In our case it reads:

$$\mathrm{Li}_2(e^{-\pi\sigma})+\mathrm{Li}_2(e^{\pi\sigma})=\frac{\pi^2}{3}-\frac{1}{2}\pi^2\sigma^2-i\pi^2\sigma \tag{5.16}$$

By using this, and the identity $\log(1-z)=i\pi+\log(z-1)$, one verifies that the difference of (5.13) and (5.14) vanishes. This ensures (5.12) is indeed analytic in σ. The same argument applies to the dependence of (5.12) on the combination $\sigma-\sigma'$.

Finally, we have to use the prescription (5.2) on the result (5.12). Let us suppose $m\pm n$ an even number. Then, we can systematically replace factors like $e^{\pm i\pi(m\pm n)}$ with $+1$. (for $m\pm n$ odd, we would have to replace $e^{\pm i\pi(m\pm n)}$ with -1, but the results are unchanged). Then as a consequence of applying the identity (5.2) to (5.12), on obtains the final result:

$$mn\vartheta(\sigma), \tag{5.17}$$

where:

$$\vartheta(\sigma)=\begin{cases}+1 & \text{if } \sigma>0\\ -1 & \text{if } \sigma<0\end{cases} \tag{5.18}$$

At this point one might wonder why we have obtained a step function when (5.12) was shown to be analytic. This is because (5.12) has branch cuts and when the prescription (5.2) is applied one evaluates jumps across the branch cuts [6]. Finally, we can plug these results back in (5.1) and, by recalling they were obtained after applying the operator $x\partial_x$, we obtain:

$$mn\vartheta(\sigma) = \mathcal{K} \int \frac{\mathrm{d}z}{z} (-\frac{z}{2}) \left(x - \frac{1}{x} \right) e^{-\frac{z}{2}(x+\frac{1}{x})} \Psi(z), \tag{5.19}$$

or, equivalently:

$$\frac{mn\vartheta(\sigma)}{\sinh \sigma} = \mathcal{K} \int \mathrm{d}z \, e^{-\frac{z}{2}(x+\frac{1}{x})} \Psi(z). \tag{5.20}$$

Here $\mathcal{K}$ stands for the overall constant we have not kept track of. By taking the inverse Laplace transform on the LHS using the tables in [18] one finally ends up with:

$$\Psi(z) = mnI_0(z)\, \mathcal{K}\,, \tag{5.21}$$

which happily, agrees with the results of sections 3 and 4.

5.2 The RR field

In this section we extend the analysis of the previous subsection to the RR case. We introduce the target space field $\Psi^R_\pm$ which carries the information of the RR closed string field sourced by the D-instanton. These are related the annulus amplitude $\mathcal{Z}^R(m,n|\sigma)$ in the Ramond sector by the following transforms

$$\mathcal{Z}^R\,(m,n|\sigma) = \int_0^\infty \frac{dz}{\sqrt{z}} e^{-z\cosh \pi\sigma} \cosh \frac{\pi\sigma}{2} \Psi^{\mathrm{RR}}_+(z), \tag{5.22}$$

where $e^{\pi\sigma} = x$. This definition of the target space field is the extension of the definition of the target space field given in (5.1) for the Ramond sector. It is motivated from the fact that the mini-superspace wave $\Psi_{p+}(z)$ in (2.3) can be written in terms of a cosine using a Backlund transform [8], which is given by

$$\frac{\pi \cos p\pi\sigma}{\cosh \pi p} = \int_0^\infty \frac{dz}{\sqrt{z}} e^{-z\cosh \pi\sigma} \Psi_{p+}(z)\,. \tag{5.23}$$

Applying $x\partial_x$ to the equation (5.22) we obtain

$$\begin{aligned} x\frac{\partial}{\partial x}\mathcal{Z}^R\,(m,n|\sigma) &= -\int_0^\infty \frac{dz}{\sqrt{z}} e^{-z\cosh \pi\sigma} \sinh \frac{\pi\sigma}{2} \left(z\frac{\partial}{\partial z} + z \right) \Psi^{\mathrm{RR}}_+(z) \\ &= \int_0^\infty \frac{dz}{\sqrt{z}} e^{-z\cosh \pi\sigma} \sinh \frac{\pi\sigma}{2} \Psi^{\mathrm{RR}}_-(z)\,. \end{aligned} \tag{5.24}$$

In the first line of the above equation we have converted the derivative ∂_σ to a differential operator in z using integration by parts. The second line of the above equation is just the definition of $\Psi^{\mathrm{RR}}_-(z)$. To proceed we need the corresponding annulus amplitude in the Ramond sector. This is obtained using the FZZT-FZZT annulus amplitude in the Ramond sector, which in turn is constructed from the RR boundary state wave-functions, $\Psi^R_\pm(p.\sigma)$,

[6] As a simple example consider $\log(1-z) - \log(1-e^{2\pi i}z)$, this vanishes for $|z| \leq 1$ and $2\pi i$ for $|z| > 1$.

in (5.3). $\Psi^R_\pm(p.\sigma)$, come in pair, and their expressions are proportional to:

$$\Psi^R_+(p,\sigma) = \cos(\pi\sigma p)\Gamma^2\left(\frac{1}{2}+ip\right)\left(\frac{\mu}{2}\right)^{-ip}, \tag{5.25}$$

and:

$$\Psi^R_-(p,\sigma) = \sin\left(\pi\sigma p\right)\Gamma^2\left(\frac{1}{2}+ip\right)\left(\frac{\mu}{2}\right)^{-ip}. \tag{5.26}$$

Case i. $m \pm n$ even. Let us begin with the case of $m \pm n$ even for the ZZ boundary state: we can obtain it from the FZZT boundary state Ψ^R_+, and as a result the ZZ-FZZT amplitude can be written in terms the FZZT-FZZT amplitude as:

$$\mathcal{Z}^R\left(m,n|\sigma\right) = \mathcal{Z}^R\left(\sigma'_{m,-n} = i(m-n)|\sigma\right) \mp \mathcal{Z}^R\left(\sigma'_{m,n} = i(m+n)|\sigma\right), \tag{5.27}$$

where the $-$ sign is for both m and n even while the $+$ sign applies to the case of both m ad n odd ($m \pm n$ is always even). The FZZT-FZZT partition function is on the other hand:

$$\mathcal{Z}^R(\sigma'|\sigma) = \int \mathrm{d}p\,\frac{\cos(p\pi\sigma')\cos(p\pi\sigma)}{\cosh^2(\pi p)}\frac{1}{p^2}. \tag{5.28}$$

After applying the operator $x\partial_x$, $x = e^{\pi\sigma}$, one finds:

$$x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = -\pi\int \mathrm{d}p\,\frac{\cos(p\pi\sigma')\sin(p\pi\sigma)}{\cosh^2(\pi p)}\frac{p}{p^2}, \tag{5.29}$$

and the relevant integral to compute is:

$$x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = -\frac{\pi}{2i}\int \mathrm{d}p\,\frac{e^{ip\pi(\sigma+\sigma')}+e^{ip\pi(\sigma-\sigma')}}{\cosh^2(\pi p)}\frac{1}{p}. \tag{5.30}$$

Again, we evaluate this integral in the complex plane; to do this we choose $\sigma+\sigma'$ and $\sigma-\sigma'$ to be positive and close the contour in the upper half plane. At the end we will show that the expression we obtain will be analytic in σ and σ'. The above integral has double poles at $p = i(k+\frac{1}{2})$ and a single pole at $p=0$ for which an ϵ-prescription is required. We use the following ϵ prescription on the integral (5.30)

$$x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = -\lim_{\epsilon\to 0}\frac{\pi}{2i}\int \mathrm{d}p\,\frac{e^{ip\pi(\sigma+\sigma')}+e^{ip\pi(\sigma-\sigma')}}{\cosh^2(\pi p)}\frac{p}{(p+i\epsilon)(p-i\epsilon)}, \tag{5.31}$$

By evaluating the residues at the poles, one finds:

$$\begin{aligned} x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = &-\frac{1}{\pi^2}\sum_{k=0}^{\infty}\frac{i\pi(\sigma'+\sigma)e^{-\pi(k+\frac{1}{2})(\sigma+\sigma')}+i\pi(\sigma-\sigma')e^{-\pi(k+\frac{1}{2})(\sigma-\sigma')}}{i(k+\frac{1}{2})} - \\ &-\frac{1}{\pi^2}\sum_{k=0}^{\infty}\frac{e^{-\pi(k+\frac{1}{2})(\sigma+\sigma')}+e^{-\pi(k+\frac{1}{2})(\sigma-\sigma')}}{(k+\frac{1}{2})^2}+1. \end{aligned} \tag{5.32}$$

The last terms arises due to the pole at $p = i\epsilon$ which is given by

$$\lim_{\epsilon\to 0}\frac{e^{-\epsilon\pi(\sigma+\sigma')}+e^{-\epsilon\pi(\sigma-\sigma')}}{(\cosh\pi i\epsilon)^2}\frac{i\epsilon}{2i\epsilon} = \frac{1}{2}+\frac{1}{2} = 1. \tag{5.33}$$

Now we resume the expressions in (5.32) using the following formulae:

$$\sum_{k=0}^{\infty} \frac{x^{(k+\frac{1}{2})}}{k+\frac{1}{2}} = -\log(1-\sqrt{x}) + \log(1+\sqrt{x}) \,, \qquad \sum_{k=0}^{\infty} \frac{x^{(k+\frac{1}{2})}}{(k+\frac{1}{2})^2} = 2\mathrm{Li}_2(\sqrt{x}) - 2\mathrm{Li}_2(-\sqrt{x}) \,. \tag{5.34}$$

Using these formulae we rewrite equation (5.32) as:

$$\begin{aligned} x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = & -\frac{1}{\pi}\Big[(\sigma+\sigma')\left(-\log\left(1-e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)+\log\left(1+e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)\right)+ \\ & +(\sigma-\sigma')\left(-\log(1-e^{-\frac{\pi}{2}(\sigma-\sigma')})+\log\left(1+e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)\right)\Big]- \\ & -\frac{2}{\pi^2}\Big[\mathrm{Li}_2\left(e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)-\mathrm{Li}_2\left(-e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)+ \\ & +\mathrm{Li}_2\left(e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)-\mathrm{Li}_2\left(-e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)\Big]+1\,. \end{aligned} \tag{5.35}$$

We now show that the expression in (5.35) is an analytic function of $\sigma+\sigma'$ and $\sigma-\sigma'$. Consider the terms in (5.35) which dependent only on $\sigma+\sigma'$ and label this combination as σ, then these terms for $\sigma > 0$ are given by

$$\begin{aligned} & -\frac{\sigma}{\pi^2}\left(-\log\left(1-e^{-\frac{\pi}{2}\sigma}\right)+\log(1+e^{-\frac{\pi}{2}\sigma})\right)- \\ & -\frac{2}{\pi^2}\left(\mathrm{Li}_2\left(e^{-\frac{\pi}{2}\sigma}\right)-\mathrm{Li}_2\left(-e^{-\frac{\pi}{2}\sigma}\right)\right)+\frac{1}{2} \end{aligned} \tag{5.36}$$

whereas for $\sigma < 0$ we have to close the contour in (5.30) on the lower half plane, this gives:

$$\begin{aligned} & -\frac{\sigma}{\pi}\left(-\log\left(1-e^{\frac{\pi}{2}\sigma}\right)+\log(1+e^{\frac{\pi}{2}\sigma})\right)+ \\ & +\frac{2}{\pi^2}\left(\mathrm{Li}_2\left(e^{\frac{\pi}{2}\sigma}\right)-\mathrm{Li}_2\left(-e^{\frac{\pi}{2}\sigma}\right)\right)-\frac{1}{2}\,. \end{aligned} \tag{5.37}$$

Here an overall minus sign appears because of the change of orientation in the integration contour and σ in (5.36) is replaced by $-\sigma$. To show is analyticity in σ we have to show that the difference of (5.36) and (5.37) vanishes. On taking the difference we have to use the following identities satisfied by the di-logarithms.

$$\mathrm{Li}_2(e^{-\frac{\pi}{2}\sigma})+\mathrm{Li}_2(e^{\frac{\pi}{2}\sigma}) = \frac{\pi^2}{3}-\frac{1}{2}\left(\frac{\pi\sigma}{2}\right)^2+i\frac{\pi^2}{2}\sigma\,. \tag{5.38}$$

This identity is obtained from (5.15) by setting $y = e^{\pi\sigma/2}$. It is also convenient to use the following identity [17]

$$\mathrm{Li}_2(-e^{-\frac{\pi}{2}\sigma})+\mathrm{Li}_2(-e^{\frac{\pi}{2}\sigma}) = -\frac{\pi^2}{6}-\frac{1}{2}\left(\frac{\pi\sigma}{2}\right)^2 \tag{5.39}$$

Using these identities one can indeed show that the the difference between (5.36) and (5.37) vanishes. The same argument applies to the terms depending on the combination $\sigma-\sigma'$ in (5.35). Therefore we have shown that the final expression in (5.35) in an analytic function in the $\sigma+\sigma'$ and $\sigma-\sigma'$.

Now we have to use the prescription given in (5.27) to finally obtain the ZZ-FZZT annulus amplitude. Using the expression in (5.35) for the FZZT-FZZT annulus amplitude in the Ramond sector we obtain

$$\begin{aligned}\frac{\partial}{\partial\sigma}\mathcal{Z}^R(\sigma'|\sigma) &= 0 \qquad m,n \quad \text{even} \\ &= \mathcal{K}\,\vartheta(\sigma) \qquad m,n \quad \text{odd}\,. \end{aligned} \tag{5.40}$$

Here $\mathcal{K}$ refers to the overall normalization which we have not kept track of and ϑ is the step function introduced in (5.18). Using the definition of Ψ_-^{RR} given in (5.24) and taking the appropriate inverse Laplace transform we obtain

$$\begin{aligned}\Psi_-^{\mathrm{RR}} &= \mathcal{K}\,e^z \qquad m,n \quad \text{odd}\,, \\ &= 0 \qquad m,n \quad \text{even}\end{aligned} \tag{5.41}$$

Case ii. $m \pm n$ **odd** To obtain the ZZ-FZZT amplitude when when $m \pm n$ is odd we have to use the FZZT boundary state wave function $\Psi_-^R(p,\sigma)$ given in (5.26). The FZZT-FZZT partition function is then written as

$$\mathcal{Z}^R(\sigma'|\sigma) = \int dp \frac{\sin(p\pi\sigma')\sin(p\pi\sigma)}{\cosh^2(\pi p)}\frac{1}{p^2}\,. \tag{5.42}$$

Given the above partition function the ZZ-FZZT amplitude is given by the identity

$$\mathcal{Z}^R(m,n|\sigma) = \mathcal{Z}^R(\sigma' = i(m+n)|\sigma) \mp \mathcal{Z}^R(\sigma' = i(m-n)|\sigma)\,, \tag{5.43}$$

where the $-$ sign is for m odd and n even while the $+$ sign is for the case of m even and n odd. Now we proceed as before by first applying the operator $x\partial_x$ with $x = e^{\pi\sigma}$ on (5.42), this gives

$$\pi x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = \pi\int dp \frac{\sin(p\pi\sigma')\cos(p\pi\sigma)}{\cosh^2(\pi p)}\frac{p}{p^2}\,. \tag{5.44}$$

We can now re-write the above integral in the convenient form given below

$$\pi x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = \frac{\pi}{2i}\int dp \frac{e^{ip\pi(\sigma+\sigma')} - e^{ip\pi(\sigma-\sigma')}}{\cosh^2(\pi p)}\frac{1}{p}\,. \tag{5.45}$$

Comparing the above expression with (5.30) we see that the only change is the presence of the relative sign between the two terms. Therefore using the same steps as discussed for the case of $m \pm n$ even, we obtain the following final formula for the ZZ-FZZT amplitude

$$\begin{aligned} x\frac{\partial}{\partial x}\mathcal{Z}^R(\sigma'|\sigma) = &-\frac{1}{\pi}\Big[(\sigma+\sigma')\left(-\log\left(1-e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)+\log\left(1+e^{-\frac{\pi}{2}(\sigma+\sigma')}\right)\right) - \\ &\qquad -(\sigma-\sigma')\left(-\log\left(1-e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)+\log\left(1+e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)\right)\Big] \\ &-\frac{2}{\pi^2}\Big[\mathrm{Li}_2\left(e^{-\frac{\pi}{2}(\sigma+\sigma')}\right) - \mathrm{Li}_2\left(-e^{-\frac{\pi}{2}(\sigma+\sigma')}\right) - \\ &\qquad - \mathrm{Li}_2\left(e^{-\frac{\pi}{2}(\sigma-\sigma')}\right) + \mathrm{Li}_2\left(-e^{-\frac{\pi}{2}(\sigma-\sigma')}\right)\Big]\,. \end{aligned} \tag{5.46}$$

From the arguments discussed for the previous case it is clear that the above expression is an analytic function of $\sigma+\sigma'$ and $\sigma-\sigma'$. To obtain the ZZ-FZZT amplitude we can now use the prescription given in (5.43). Substituting the expression (5.46) for the ZZ-FZZT amplitude we see that the

$$x\frac{\partial}{\partial x}\mathcal{Z}^R(m,n|\sigma)=0\,,\qquad (m\pm n)\quad \text{odd}\,. \tag{5.47}$$

Therefore we obtain the result that the RR charge by a (m,n) D-instanton vanishes unless m,n are both odd and is equal to that of the $(1,1)$ D-instanton.

6. Conclusions

In this paper we have used three methods to study the behaviour of the closed string fields $\Psi(z)$ sourced by (m,n) D-instantons of type 0B theory distributed uniformly in the time direction. The three approaches were: the mini-superspace method, the extension of wave-functions beyond the mini-superspace approximation and finally using the ZZ-brane as a probe of the closed string field. In all the three methods we obtained the tension of an $(m,n)=(t,1)$ brane is proportional to $mn=t$ times the tension of the $(1,1)$ brane and the RR scalar charge of these branes is non-zero only for the case of t odd and is equal to that of the $(1,1)$ brane. As a further consistency check, we verified that the closed string fields sourced by the D-instantons satisfied the corresponding equations of motions. The consistency of these results lends support to the three methods. In fact we find the functional dependence of the closed field from the mini-superspace approach and that determined by using the ZZ-probe method agrees identically. It will be interesting to provide a reason for this, because in principle the exact closed string wave-functions and the equations of motion can be different from the mini-superspace limit. These results also imply that that the $(t,1)$ D0-branes of type 0A theory have the same property as these are T-dual to the corresponding D-instantons considered here. These branes will carry RR one form charge of type 0A if t is odd and will be equal to that of the $(1,1)$ D0-brane. At this point it is worth to make a comment about the relation between our results and a result obtained in [14]. There, the minimal (p,q) string theory was studied, for which $b^2=p/q$ is rational. In this case the inequivalent ZZ boundary states are labelled by $(t-1,m,n)$ with $t\geq 1$, $0<n\leq p$ and $0<m\leq q$. In [14] the boundary state corresponding to $(t-1,m,n)$ was shown to be t times the boundary state corresponding to $(0,m,n)$. Setting formally $p=q=1$, so that $m=n=1$, to recover the $c=1$ case, this gives the result that the tension of $(t-1,1,1)$, which coincides with the state we labelled by $(t,1)$, goes like t, which formally agrees with our results. However the result of [14] is a consequence of a peculiar shift symmetry $\sigma\to\sigma+2i\sqrt{pq}$ in the FZZT boundary states which are used to construct the ZZ boundary states. This is due to the fact that in the minimal string case the closed string states in the BRST cohomology involve only discrete Liouville states. This is to be contrasted with our case, where the matter is a $c=1$ system and the BRST cohomology involves also continuum, non-degenerate Liouville states, so that the above symmetry is not obvious.

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The fact the tension of the $(t, 1)$ brane increases as t and the RR scalar charge is just unity or zero indicates that these are unstable. This is in accordance with the presence of many tachyons in the open string spectrum of these branes. From examining the annulus amplitude of two $(t, 1)$ D-instantons with the appropriate GSO projection for type 0B theory it is easy to see that there are roughly of the order of t tachyons.[7] It is an open question to study the condensation of these tachyons, the height of the potential we now know is proportional to t. In the matrix formulation of type 0B theory the $(1, 1)$ ZZ brane plays a special role and is believed to be fundamental. Since the $(t, 1)$ branes are not stable for $t > 1$, it will be interesting to see if they can be thought of as excited states made of $(1, 1)$ and anti-$(1, 1)$ branes. Finally, the (m, n) ZZ boundary state can be constructed for all values of b, in this paper we studied the $b \to 0$ limit and the point $b = 1$. We have seen that for both these cases the tension and RR charge of these branes were same. It will be of interest to see if the results of the tension and the RR scalar charge of these branes depend on the parameter b

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